Exercise 2.4.4

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

$$\dot{x} = x^2(6 - x)$$

Solution

The fixed points occur where $\dot{x} = 0$.

$$x^{*2}(6 - x^*) = 0$$

$$x^{*2} = 0 \quad \text{or} \quad 6 - x^* = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = 6$$

Use linear stability analysis to classify these points.

$$\begin{aligned} f(x) &= x^2(6 - x) \\ &= 6x^2 - x^3 \end{aligned}$$

Differentiate $f(x)$.

$$f'(x) = 12x - 3x^2$$

As a result,

$$\begin{aligned} f'(0) &= 0 && \Rightarrow \quad \text{No conclusion can be made about the stability of } x^* = 0. \\ f'(6) &= -36 < 0 && \Rightarrow \quad x^* = 6 \text{ is a stable fixed point.} \end{aligned}$$

The graph of $\dot{x}$ versus x below shows that $x^* = 0$ is in fact a half-stable point and that the second result is true.